On the Criterion of Ordinary Hydrodynamic Instability in Topheavy Fluid Layers

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INTRODUCTION

The motion of elements in a fluid layer subject to forces is divisable into a general systematic one and into another one due to eddies of differing sizes subject to random fluctuations. Convective motion takes place if one, or more, of the following conditions is satisfied:

a) The lower surface of a fluid layer is heated relative to the top layer.

b) The top layer of the fluid is cooled relative to the lower layer.

c) A fluid of relatively less density is injected at lower levels.

d) One, or more, of the above conditions is combined with Coriolis forces or a magnetic field in case of an ionized fluid.

The finite values of viscosity, conductivity or diffusivity of a fluid allow topheaviness to be maintained before the onset of convective instability. When the temperature height distribution under steady conditions is not linear, as in a gas layer, some remarks are called for. This is attempted here.

HISTORICAL

Early in this century, Benard (1901) showed that when the lower surface of a thin viscous liquid layer was heated, convective instability set in only after the temperature difference between the lower and upper surfaces exceeded a limiting value.

Aichi (1907) and Rayleigh (1916) derived a quantitative criterion of the possible top heaviness that could be maintained in a thin viscous liquid layer heated from below before the onset of instability based on the hydrodynamical equations of motion of the fluid layers, the equation of continuity, the equation of state and the equation of heat transport in liquids following Boussinesq. Under steady

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ABSTRACT

Early in this century, Benard found that thin, viscous liquid layers heated from below broke up into almost regular polygonal cells. From earlier theoretical investigations, expressions were obtained for the critical temperature gradients, it being assumed constant with height, for the onset of instability. Measurements in the atmosphere indicated that the temperature differences between layers near the ground often attain several times the maximum predicted temperature differences. This could be partly accounted for by the use of nonlinear (hyperbolic sine) temperature distribution as found by measurements. The temperature height distribution could in turn be explained in terms of radiative heat transfer due to layers of water vapour in the atmosphere. The corresponding problems with exponential and sinusoidal temperature distributions have now been solved. Explicit criteria for the onset of convective instability have been obtained. The critical temperature difference is larger (smaller) if the temperature - height curve is concave (convex) upwards.

conditions, the temperature decreased linearly with height in the liquid. Rayleigh reduced the problem to the solution of a single sixth order linear differential equation with constant coefficients. Under boundary conditions: both top and bottom <u>free,</u> he derived that $R = \beta g p c_p T^4 / k V \theta_o \pi^4 < \frac{27}{4}$ (i)

where p is the density of the liquid,

c_p the specific heat at constant pressure,

k the coefficient of molecular conductivity,

v the coefficient of molecular viscosity,

T the thickness of the fluid layer,

θ_o a constant reference temperature,

g the value of gravity, and

β the negative gradient of temperature with height in the liquid layer (the value of β is taken as positive). In a liquid layer, $1/\theta_o$ is replaced by the coefficient of thermal expansion.

The actual values of viscosity and conductivity are very small in air that for criteria there, larger corresponding entities based on eddy considerations (eddy conductivity or diffusivity) by G.I. Taylor (1915) have been used. Low and Brunt (1925) applied the criterion to derive the maximum possible temperature gradient with height in the lower atmosphere particularly near the ground.

Jeffreys (1926, 1928) solved the sixth order differential equation with both the boundaries as <u>rigid</u>. In his later paper (1928) he stated that a direct substitution of a Fourier expansion in the differential equation led to difficulties and adopted an <u>ad hoc</u> method of solution. Low (1929) factorised the differential operator of the equation considering it as a sum of two cubes. He considered also the case when one boundary was <u>free</u> and the other <u>rigid</u>.

Top heaviness in a liquid layer could be had by pouring a highly volatile liquid over a less volatile but more viscous liquid layer. The top surface would then be cooled relative to the bottom one.

The number of sides of the possible regular polygonal cells in which the liquid sheet breaks up, due to convective instability, had several times been given as 3 to 7. on grounds of symmetry, only regular hexagons, squares, equilateral triangles and parallel bands are only possible. (Malurkar, 1936, 1937a).

Experimentally, from about 1928, creation of polygonal cells in liquid mixtures was once again undertaken at the Imperial College of Science and Technology (London) by Gilbert T. Walker and his collaborators to find analogies to cloud patterns. Later, David Brunt and his co-workers in the same College attempted verifications of theories with a view to meteorological applications. Schmidt and Milverton (1935), also in the same College, carried out elaborate experimental confirmations in the Engineering Section.

In discussing 'Dynamics of Thunderstorms', the author (1937a, 1943) tried to show that the convective instability brought about by uniformly stratified top heaviness could not give rise to the instability needed in thunderstorms (as used to be stated often potentially colder air being superposed on potentially warmer air initiating and maintaining thunderstorms). In that discussion, the top heaviness brought about by the superposition of fluid layers of differing density was briefly considered (e.g. a dry stream over a moist one). The thermal structure was taken for simplicity as isothermal. The equation of heat transport was replaced by the diffusion equation (Fick's). The density variation with height was taken as linear. The variations in viscosity were ignored to a first approximation. The criterion of convective instability could be expressed in terms of the difference of density of top and bottom layers almost similar to Rayleigh criterion above.

ROLE OF RADIATION EFFECT

The actual temperature gradient with height near the ground on sunny days was much more than the limits obtained above. Among the additional hypothesis, the one about the modification introduced by radiative transfer seemed promising. George C. Simpson (1928) had attempted to fit numerically a temperature height curve of the atmosphere by taking account of the effect of the stratified layers of water vapour on the radiative transfer of heat radiation. Brunt (1929, 30) considered the modification, due to the effect of water vapour in layers, on the equation of heat transport. He added to the eddy diffusivity term in the Taylor type equation, one due to radiation ($K_E + K_R$ instead of K_E). The form of the equation of heat transfer remained unchanged. Under steady conditions, the temperature height curve remained linear.

But the temperature measurements, on a hot sunny afternoon near the ground (flat surface) which was producing inferior mirages showed that in about the first twenty centimeter layer, it decreased with height as a hyperbolic sine curve. At similar height intervals, above that height, the temperature could be considered as almost constant. (Malurkar and Ramdas, 1932). The temperature height curve over a heated flat plate (Ramdas and Malurkar, 1932) was also found to be similar. In the small height interval where the mean temperature gradient was large, the difference between the bottom and the top layers was ten or more times the expected maximum value given by Brunt (1930). Malurkar and Ramdas (1932) advanced an explanation for the hyperbolic sine temperature height curve by assuming that heat radiation was absorbed and emitted in successive layers of water vapour near the ground. The term due to radiation was found to be distinct and could not be added to the heat diffusivity coefficient. Roberts (1929-30) had earlier shown that that the term due to radiation was not similar to one due to heat diffusivity.

The radiative effects depend on the temperature and absorptive contents of even distant layers while the diffusivity or heat conductivity effects depend on local gradients of temperature and of density. Treating the effects of radiation and of heat diffusion as of distinct types, a small temperature disturbance at a lower level would get damped in its progressive movement upwards (Malurkar, 1934).

Under steady conditions, the modifications could explain a hyperbolic sine temperature height cruve. The nature of the temperature height curve and the maximum possible temperature difference that could be maintained in a top heavy layer of fluid for a given temperature height curve could be treated as two independent parts of a problem. (Malurkar, 1937c). In the equation of heat transport, the addition of terms due to radiative effect, could from a purely mathematical point of view be considered even as empirical additions from observations. The validity of the explanation of the temperature height curve would not prejudice the derivation of the possible maximum temperature difference between the two layers.

Digressing slightly; if, in layer of fluid, the temperature height curve (φ, Z) is concave upwards, the value of $\frac{d^2\varphi}{dZ^2}$ is positive. Then under steady conditions, the temperature at every level is lower than in the linear case. It shows that less heat has been carried up from the source at the bottom to every higher level or that more heat is transferred or dissipated away from every level than in the case of a linear curve. A possible mechanism which could carry away heat from every layer might be radiation from successive levels resulting in a net loss of heat. It is conceivable that there might be rare instances when there is a net gain of heat in every layer due to some process. Then the temperature height curve would be convex upwards and $\frac{d^2\varphi}{dZ^2}$ negative.

Small z/u.

Small z/u.

While in the case of liquids, the temperature height distribution is linear, in the case of gas layers it would be too much of a restriction. It would be useful to obtain criteria of convective instability of fluids when under steady conditions, the temperature height curve is concave or convex upwards.

RESUME OF THE HYPERBOLIC SINE CURVE CASE

Small z

If Z is the height measured upwards, the temperature height structure for the varying part was taken as

$$\frac{\beta T \sinh \alpha\,(T-Z)}{\sinh \alpha T} \quad \text{or} \quad \frac{\beta T \sinh \gamma\xi}{\sinh \pi\gamma}$$

Small z pi.

if $\pi\,(T-Z) = T\xi$ and $\pi\gamma = \alpha T$ . . . (ii)

β is now the mean negative gradient of temperature between 0 and T, and α^2 is a constant. The resulting basic differential equation was: if $b^2 = \alpha^2 + \gamma^2$

$$(d^2/d\xi^2 - a^2)^2 \operatorname{sech} \gamma\xi\,(d^2/d\xi^2 - b^2)\,\chi = -Ra^2\pi r \operatorname{cosech} \pi r\,\chi \ldots \qquad \text{(iii)}$$

where χ was a function of the temperature, a^2 was a constant and other symbols have been explained earlier.

The equation simplifies to those of Aichi, Rayleigh and others if $\gamma = 0$. The methods of solution adopted by Jeffreys and Low for linear differential equations with constant coefficients could not be extended easily. Malurkar (1937b) formulated, for the first time, the use of a series solution for the type of equations arising in stability problems. Instead of a direct substitution of an expansion in a set of orthogonal functions in the differential equation, the latter was split up into two suitable parts and the series expansion was substituted in only one part. The resulting differential equation was completely solved taking account of boundary conditions. A comparision of this solution with the initial assumption led to an explicit determination of all the coefficients. Malurkar and Srivastava (1937) verified that the method led to earlier results by solving a slightly generalised Rayleigh-Jeffreys equation.

The above led to a consistency condition of an infinite determinant being equated to zero. The value of Ra^2 could be obtained by successive approximations. To a first approximation, the smallest value of Ra^2 in the case when both boundary surfaces are taken as <u>free</u> (Rayleigh type), leads to the maximum temperature difference criterion:

$$R = \beta g p c_p T^4 / k v \theta_0 \pi^4 < \frac{(1+a^2)^3}{a^2} \left\{ 1 + \frac{(5+a^2)}{4(1+a^2)} \gamma^2 \right\} \ldots \qquad (4)$$

with $a^2 = \frac{1}{2}$ (see later for the minimal value) the above is reduced to

$\frac{27}{4} (1 + \frac{11}{12} \gamma^2)$. Even for other types of boundary conditions, the criteria of maximum possible temperature difference increased.

Considerable work on fluid instability has appeared in its various aspects in recent years. A reference to later memoirs and treatises (Sutton, 1950; Backus, 1955; Lin, 1955; Chandrasekhar, 1961) may be made. However, the equation of heat transport has been nearly the same as that given by Rayleigh.

EXPONENTIAL TEMPERATURE - HEIGHT CURVE

The same equation of heat transport used for the hyperbolic sine temperature height decrease is valid also for the exponential temperature height decrease case. Such exponential temperature decrease with height can occur in geophysical problems.

EQUATIONS

Take xyz as a rectangular system of coordinates with z axis directed upwards; u, v, and w as components of velocity in a fluid layer; ρ as the density of the fluid; p as the pressure at xyz; g as the value of gravity;

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ρ

ν as the kinematic coefficient of viscosity and ν' as the second coefficient of viscosity. The equations of motion can be written as:

$$\rho\, {}^{d}/_{dt}\,(u,v,w) - \nu\rho\nabla^2(u,v,w) - ({}^{\nu}/_{3} + \nu')\,\rho\,({}^{\partial}/_{\partial x}, {}^{\partial}/_{\partial y}, {}^{\partial}/_{\partial z})\,({}^{\partial u}/_{\partial x} + {}^{\partial v}/_{\partial y} + {}^{\partial w}/_{\partial z}) = -\,({}^{\partial}/_{\partial x}, {}^{\partial}/_{\partial y}, {}^{\partial}/_{\partial z})\,p - (o,\ o,\ \rho g) \qquad (1)$$

where $\nabla^2 = {}^{\partial^2}/_{\partial x^2} + {}^{\partial^2}/_{\partial y^2} + {}^{\partial^2}/_{\partial z^2}$

The equation of continuity is

$$1/\rho\ {}^{d\rho}/_{dt} + ({}^{\partial u}/_{\partial x} + {}^{\partial v}/_{\partial y} + {}^{\partial w}/_{\partial z}) = 0$$

If E be the heat energy that crosses a unit layer at xyz apart from that due to heat conductivity or diffusivity, and if φ be the temperature in excess of a given constant one θ_0 (attained above a certain height), the equation of heat transport can be:

$$\rho c_p\, {}^{d\varphi}/_{dt} = k\nabla^2\varphi - ({}^{\partial E}/_{\partial x} + {}^{\partial E}/_{\partial y} + {}^{\partial E}/_{\partial z}) \qquad (3)$$

If the total variation of temperature in the layer is small compared with the actual magnitude of temperature involved, this equation can be simplified to a good approximation, to

$$\rho\, c_p\ {}^{d\varphi}/_{dt} = k(\nabla^2\varphi - \alpha^2\varphi) \qquad (4)$$

where α^2 is a constant. (Malurkar and Ramdas, 1932; and Malurkar, 1932).

If the variations of pressure in the layer are small enough to be neglected, the equation of state would be: $\rho\,(\theta_0 + \varphi) = \rho_0\theta_0$ (5)

Apart from that due to this temperature effect, the density variations in the layer become and are taken here as negligible.

Assuming that steady conditions exist, the equations (1) to (4) reduce to:

$$\partial\rho_i/_{\partial z} + g\rho_i = 0 \text{ and } \partial^2\varphi_i/_{\partial z^2} = \alpha^2\varphi_i \qquad (6)$$

where i is the subscript denoting steady conditions. Let $\rho = \rho_i + \rho'$; $p = p_i + p'$ etc. Neglecting the squares and products of small quantities (departures from steady conditions), the above can be simplified as before (Malurkar, 1937c).

$$\partial\varphi/\partial t + w\,\partial\varphi/\partial z = k(\nabla^2-\alpha^2)\varphi \quad \cdots (7)$$

and

$$\nabla^2\left(\nabla^2 - \frac{1}{\nu}\partial/\partial t\right)\left\{\frac{1}{\partial\varphi/\partial z}\left(\nabla^2-\alpha^2-\frac{\rho c_p}{k}\partial/\partial t\right)\varphi\right\} + \frac{g\rho c_p}{k\nu\theta_0}\left(\partial^2\varphi/\partial x^2+\partial^2\varphi/\partial y^2\right) = \left[-\frac{(\nu+3\nu')}{3\nu\theta_0}\nabla^2(\nabla^2-\alpha^2)\partial\varphi/\partial z + \frac{1}{3\nu\theta_0}\left\{(4\nu+3\nu')\nabla^2-3\partial/\partial t\right\}(\nabla^2-\alpha^2)\partial\varphi/\partial z\right] \quad \cdots (8)$$

For the present, the terms on the right hand side of equation (8) can be neglected in comparision with the other ones. Hence equation (8) becomes

$$\nabla^2\left(\nabla^2 - \frac{1}{\nu}\partial/\partial t\right)\left\{\frac{1}{\partial\varphi/\partial z}\left(\nabla^2-\alpha^2-\frac{\rho c_p}{k}\partial/\partial t\right)\varphi\right\} + \frac{g\rho c_p}{k\nu\theta_0}\left(\partial^2\varphi/\partial x^2+\partial^2\varphi/\partial y^2\right) = 0 \quad \cdots (9)$$

If there be a transition from stable to unstable conditions, it should occur when $\delta/\delta t = 0$ in the above and hence

$$\nabla^4\left\{\frac{1}{\partial\varphi/\partial z}(\nabla^2-\alpha^2)\varphi\right\} + \frac{g\rho c_p}{k\nu\theta_0}\left(\partial^2\varphi/\partial x^2+\partial^2\varphi/\partial y^2\right) = 0 \quad \cdots (10)$$

To the same order of approximation, the steady state of value of $\partial\varphi/\partial z$ could be substituted in the denominator of the first term.

Assume, as usual, that $\varphi' = \chi\varepsilon^{i(lx+my)}$ where $\varphi' = \varphi - \varphi_i$. Then

$$\partial^2\varphi_i/\partial x^2 = 0 = \partial^2\varphi_i/\partial y^2 \;;\; \partial^2\varphi_i/\partial z^2 = \alpha^2\varphi_i$$

and

$$\left(d^2/dz^2 - l^2 - m^2\right)^2\left\{\frac{1}{d\varphi_i/dz}\left(d^2/dz^2 - l^2 - m^2 - \alpha^2\right)\chi\right\} = \frac{g\rho c_p}{k\nu\theta_0}(l^2+m^2)\chi \quad \cdots (11)$$

The temperature height curve in case of an exponential decrease of temperature with height can be taken as $\varphi i = \beta T \varepsilon^{\alpha(T-Z)}/(\varepsilon^{\alpha T}-1)$ where T is the thickness or depth of the fluid layer and β is the mean negative gradient of temperature with height.

Small z

Choosing a new variable $T\xi=\pi(T-Z)$ and with the notation $\pi r = \alpha T$; $\pi^2 a^2 = T^2(\ell^2+m^2)$ and $b^2 = a^2 + r^2$ the basic equation of the problem reduces to:

Greek r

$$(d^2/d\xi^2 - a^2)^2 \varepsilon^{-r\xi}(d^2/d\xi^2 - b^2)\chi = -Ra^2\pi r\chi/(\varepsilon^{\pi r}-1) \quad \ldots (12)$$

where $R = \beta g p c_p T^4/k\nu\theta_o\pi^4$.

As the gradient of temperature with height is not constant, this linear differential equation has also not got constant coeeficients. The method of solution follows that used earlier. (There are several misprints on p. 274 and p. 275 in Malurkar, 1937c; but htye have not been carried over in later pages.)

In the interval ~~$0<\xi<\pi$~~ $0<\xi<\pi$, assume that $\chi =$ ~~$\sum_{s=1} P_s \sin s\xi$~~ $\sum_{s=1} P_s \sin s\xi$ $\quad \ldots (13)$

Substitute this series for χ only on the right hand side of equation (12). Then

$$(d^2/d\xi^2 - a^2)^2 \varepsilon^{-r\xi}(d^2/d\xi^2 - b^2)\chi = -Ra^2\pi r/(\varepsilon^{\pi r}-1)\cdot\left\{\sum P_s \sin s\xi\right\} \quad \ldots (14)$$

This equation is solved, by taking initially

$\psi\varepsilon^{r\xi} = (d^2/d\xi^2 - b^2)\chi$ which could be written as

$$= 2\{A_0 + B_0(\pi/2-\xi)\}\varepsilon^{r\xi}\cosh a(\pi/2-\xi) + 2\{A_1 + B_1(\pi/2-\xi)\}\varepsilon^{r\xi}\sinh a(\pi/2-\xi) - Ra^2\pi r/(\varepsilon^{\pi r}-1)\cdot\left\{\sum \frac{P_s\,\varepsilon^{r\xi}\sin s\xi}{(s^2+a^2)^2}\right\} \quad \ldots (15)$$

where Ao, Bo, A_1 and B_1 are constants depending on particular boundary conditions. Though the method of solution could be used to solve explicitly for χ, the criterion of convective instability under usual type of boundary conditions could be had from the above steps. The boundary conditions for χ get incorporated without its being evaluated explicitly. The usual boundary conditions are:

(a) χ, Ψ, $d^2\Psi/d\xi^2 = 0$ at $\xi = 0$ and at $\xi = \pi$ (Both Free. Rayleigh type).

(b) χ, Ψ, $d\Psi/d\xi = 0$ at $\xi = 0$ and at $\xi = \pi$ (Both Rigid. Jeffreys type).

(c) χ, Ψ, $d^2\Psi/d\xi^2 = 0$ at $\xi = 0$ and χ, Ψ, $d\Psi/d\xi = 0$ at $\xi = \pi$ (One Free and the other Rigid. Low and Chandra-~~xx~~sekhar type).

With the four common boundary conditions and with two constants A and B; equation (15) can be written as:

$$(d^2/d\xi^2 - b^2)\chi + Ra^2 r^2 \sum_n a_n \sin n\xi$$
$$= 2 \sinh a\pi/2 \{\pi A/2 - B(\pi/2 - \xi)\} \epsilon^{r\xi} \cosh a(\pi/2 - \xi)$$
$$+ 2 \cosh a\pi/2 \{\pi B/2 - A(\pi/2 - \xi)\} \epsilon^{r\xi} \sinh a(\pi/2 - \xi) \quad \cdots (17)$$

where

$$a_n = \sum_{s=1} \frac{4 n s P_s (\epsilon^{\pi r} \cos n\pi \cos s\pi - 1)/(\epsilon^{\pi r} - 1)}{\{(n^2 - s^2 + r^2)^2 + 4 r^2 s^2\}(s^2 + a^2)^2}$$

Comparing the coefficients of Sin $n\xi$. it follows that:

$$- P_n / Ra^2 + r^2(n^2+b^2)Q_n =$$

$$\frac{2n}{\pi(n^2+b^2)Ra^2}\Bigg[$$

$$\frac{2\pi a r\{B(1-\cos n\pi\, \varepsilon^{\pi r}) - A(1+\cos n\pi\, \varepsilon^{\pi r})\}}{\{(n^2+b^2)^2 - 4a^2r^2\}}$$

$$- \sinh a\pi \{B(1-\cos n\pi\, \varepsilon^{\pi r}) + A(1+\cos n\pi\, \varepsilon^{\pi r})\} \times$$

$$\left\{\frac{(r-a)}{(n^2+b^2-2ar)^2} + \frac{(r+a)}{(n^2+b^2+2ar)^2}\right\}$$

$$- \{B(\cosh a\pi - 1)(1+\cos n\pi\, \varepsilon^{\pi r}) + A(\cosh a\pi + 1)(1-\cos n\pi\, \varepsilon^{\pi r})\} \times$$

$$\left\{\frac{(r-a)}{(n^2+b^2-2ar)^2} - \frac{(r+a)}{(n^2+b^2+2ar)^2}\right\}\Bigg] \quad \ldots (19)$$

The other additional boundary conditions of equation (16) lead to an infinte number of equations linear in P_n's. An infinite determinant which has to vanish follows as a condition of consistency. The elements of the determinant may be denoted as $\mathcal{T}_{ns}$ when $n \neq s$. The diagonal elements would then be $\mathcal{T}_{nn} - 1/Ra^2$. The values of $\mathcal{T}_{ns}$ for different boundary conditions are stated here:

When both the boundaries are taken as rigid (Jeffreys type), the value of $\mathcal{J}_{ns}$ is given by the equation :

$$(n^2+b^2)(s^2+a^2)^2\mathcal{J}_{ns} =$$

$$\frac{4nsr^2(\varepsilon^{\pi r}\cos n\pi\cos s\pi-1)/(\varepsilon^{\pi r}-1)}{(n^2-s^2+r^2)^2+4r^2s^2}$$

$$+\frac{rns(1-\cos s\pi)/(\varepsilon^{\pi r}-1)}{(\sinh a\pi+a\pi)}\Bigg[\frac{2\pi ar(1+\cos n\pi\,\varepsilon^{\pi r})}{\{(n^2+b^2)^2-4a^2r^2\}}$$

$$+(\cosh a\pi+1)(1-\cos n\pi\,\varepsilon^{\pi r})\left\{\frac{(r-a)}{(n^2+b^2-2ar)^2}-\frac{(r+a)}{(n^2+b^2+2ar)^2}\right\}$$

$$+\sinh a\pi(1+\cos n\pi\,\varepsilon^{\pi r})\left\{\frac{(r-a)}{(n^2+b^2-2ar)^2}+\frac{(r+a)}{(n^2+b^2+2ar)^2}\right\}\Bigg]$$

$$-\frac{rns(1+\cos s\pi)/(\varepsilon^{\pi r}-1)}{(\sinh a\pi-a\pi)}\Bigg[\frac{2\pi ar(1-\cos n\pi\,\varepsilon^{\pi r})}{\{(n^2+b^2)^2-4a^2r^2\}}$$

$$-(\cosh a\pi-1)(1+\cos n\pi\,\varepsilon^{\pi r})\left\{\frac{(r-a)}{(n^2+b^2-2ar)^2}-\frac{(r+a)}{(n^2+b^2+2ar)^2}\right\}$$

$$-\sinh a\pi(1-\cos n\pi\,\varepsilon^{\pi r})\left\{\frac{(r-a)}{(n^2+b^2-2ar)^2}+\frac{(r+a)}{(n^2+b^2+2ar)^2}\right\}\Bigg] \quad \ldots (20)$$

When both the boundaries are taken as free (Rayleigh type), the terms become simple and

$$\mathcal{J}_{ns}=\frac{4nsr^2(\varepsilon^{\pi r}\cos n\pi\cos s\pi-1)/(\varepsilon^{\pi r}-1)}{(n^2+b^2)\{(n^2-s^2+r^2)^2+4r^2s^2\}(s^2+a^2)^2} \quad \ldots (21)$$

When both boundaries are taken as free (Rayleigh type), the terms become simple and

When the top boundary is taken as free and the lower one as rigid (Low and Chandrasekhar type) then

$$(n^2+b^2)(s^2+a^2)^2 J_{ns} =$$

$$\frac{4nsr^2(\varepsilon^{\pi r}\cos n\pi\cos s\pi-1)/(\varepsilon^{\pi r}-1)}{\{(n^2-s^2+r^2)^2+4r^2s^2\}}$$

$$+\frac{rns\cos s\pi\sinh^2 a\pi/2}{(\varepsilon^{\pi r}-1)(a\pi\cosh a\pi-\sinh a\pi)}\Bigg[\frac{2\pi a r(1+\cos n\pi\,\varepsilon^{\pi r})}{\{(n^2+b^2)^2-4a^2r^2\}}$$

$$+(\cosh a\pi+1)(1-\cos n\pi\,\varepsilon^{\pi r})\left\{\frac{(r-a)}{(n^2+b^2-2ar)^2}-\frac{(r+a)}{(n^2+b^2+2ar)^2}\right\}$$

$$+\sinh a\pi(1+\cos n\pi\,\varepsilon^{\pi r})\left\{\frac{(r-a)}{(n^2+b^2-2ar)^2}+\frac{(r+a)}{(n^2+b^2+2ar)^2}\right\}\Bigg]$$

$$-\frac{rns\cos s\pi\cosh^2 a\pi/2}{(\varepsilon^{\pi r}-1)(a\pi\cosh a\pi-\sinh a\pi)}\Bigg[\frac{2\pi a r(1-\cos n\pi\,\varepsilon^{\pi r})}{\{(n^2+b^2)^2-4a^2r^2\}}$$

$$-(\cosh a\pi-1)(1+\cos n\pi\,\varepsilon^{\pi r})\left\{\frac{(r-a)}{(n^2+b^2-2ar)^2}-\frac{(r+a)}{(n^2+b^2+2ar)^2}\right\}$$

$$-\sinh a\pi(1-\cos n\pi\,\varepsilon^{\pi r})\left\{\frac{(r-a)}{(n^2+b^2-2ar)^2}-\frac{(r+a)}{(n^2+b^2+2ar)^2}\right\}\Bigg] \quad \cdots (22)$$

The consistency infinite determinants vanish only for discrete values of Ra^2; which, in any particular case, could be got by successive approximations. The elements of the determinant decrease rapidly with increase of n or of s. The working would not be prohibitive. As in the case of the hyperbolic sine temperature height curve, the determinant breaks up when $r = 0$, corresponding to the odd and even solutions. The smallest root of Ra^2 has to be chosen, and in the degenerate case when $r = 0$, the odd solution has to be selected. The non-linearity of the temperature height curve removes a part of the degeneracy (a type of symmetry) in the stability configurations. This would have to be dealt with separately.

special little "degeneracy"

In the particular case of both boundaries being taken as <u>free</u> (Rayleigh type), the diagonal elements of the infinite determinant have <u>the same form</u>, whether the temperature hegiht decrease follows the hyperbolic sine or exponential curve:

$$\beta T \, \frac{\sinh \pi r(1-z/T)}{\sinh \pi r} \quad \text{or} \quad \beta T \, \frac{\epsilon^{\pi r(1-z/T)}}{(\epsilon^{\pi r}-1)}$$

However, for the same difference of temperature between the top and bottom boundaries and for the same depth or thickness of the fluid layer, the value of r would not be the same in the two cases. Under the above conditions, the criteria of the maximum possible temperature difference between the lower and upper boundaries of the fluid have, to a first approximation, neglecting r^4, the same form for the exponential and the hyperbolic sine temperature height curves given by:

$$Ra^2 = (1+r^2/4)(1+a^2+r^2)(1+a^2)^2 \text{ or } (1+a^2)^3\{1+(5+a^2)r^2/4(1+a^2)\} \qquad (23)$$

The value to be taken for a^2 is $1/2+1/6r^2+0(r^4)$ leading to $R = \frac{27}{4}\left(1+\frac{11r^2}{2}\right)$ neglecting r^4. The value of R remains unchaged, to a first approximation, whether the value of a^2 (characterising the pattern in the horizontal plane) is taken from the constant gradient of temperature with height or with the exponential and the hyperbolic sine temperature height curves. It follows that while the non-linearity of the above temperature height curves modify the norizontal patterns and the maximum possible temperature difference that could be maintained, the further effect of the change in the horizontal pattern on the possible maximum temperature difference can to a first approximation be neglected. Similar results hold good for other types of boundary conditions.

For the other two types of boundary conditions also, the increase in the criterion of the maximum temperature difference between the top and bottom layers of a top heavy fluid structure would exist. While calculations have been shown only for the heperbolic sine and the exponential temperature height cruves, it would be apparent that these represent curves which are concave upwards. It can be stated that the criterion of the maximum possible temperature difference between the bottom and top layers of a top heavy fluid layer increases when the temperature height cruve is concave upwards, from the value for a linear temperature height structure. The increase in the value of the maximum possible temperature difference would be an increasing function of the departure from linearity of the temperature height curves, considering those of <u>any one</u> family.

TEMPERATURE HEIGHT CURVE CONVEX UPWARDS

A representative case of a temperature height decrease curve being convex upwards can be taken with the variational part of the temperature as

Greek "lambda" "gamma"

$$\beta T \frac{\sin \pi\gamma(1-z/T)}{\sin \pi\gamma} \qquad (24)$$

in the interval $0 < Z < T$. The corresponding equation of heat transport would have to be taken as:

$$\rho c_p \, d\varphi/dt = k(\Delta^2 + \frac{\pi^2\gamma^2}{T^2}) \varphi \qquad \nabla/\lambda \qquad (25)$$

where r^2 is a constant restricted to only small values: $\gamma^2 << \frac{1}{2}$

greek letter "gamma"

Following exactly as before, with a similar notation, the basic equation could be written as:

$$(d^2/d\xi^2 - a^2)^2\{\sec \gamma\xi(d^2/d\xi^2 - a^2 + \gamma^2)\chi\} = - Ra^2\pi\gamma \operatorname{cosec} \pi\gamma.\chi \tag{26}$$

The equation is solved as before in stages assuming $\chi = \sum_s P_s \sin s\xi$ and substituting it only on the right hand side of the equation (26) i.e.

$$(d^2/d\xi^2-a^2)^2\{\sec \gamma\xi(d^2/d\xi^2-a^2+\gamma^2)\chi\} = - Ra^2\pi\gamma\operatorname{cosec}\pi\gamma\left[\sum_s P_s \sin s\xi\right] \tag{27}$$

greek letter gamma

The solution of this equation follows analogous steps to that in the hyperbolic sine (Malurkar, 1937c) and the exponential curve cases.

It is sufficient to point out that a first approximation with both boundaries <u>free</u> (Rayleigh type), correspodning criterion would be given by:

$$R = \frac{(1+a^2)^3}{a^2} \{1-(5+a^2)\gamma^2/4(1+a^2)\} \tag{28}$$

or with $a^2 = 1/2$ by ~~$R = \frac{27}{4}(1-\frac{11}{12}r^2)$~~ $R = \frac{27}{4}(1-\frac{11}{12}r^2)$

The criterion of maximum possible temperature difference between the bottom and top layers is less than in the constant temperature gradient case. The possibility of the temperature height curve being convex upwards would be ver rare.

Hence, the criterion of instability of a layer of heated fluid, particularly in the case of gasses, depends on the temperature height curve in the fluid. While the temperature difference between the bottom and top layers might be within the limits of the criteria of stability for one temperature height structure, it might not be so for another one and vice versa.

Hales (1935) determined the criterion of instability for a compressible atmospheric layer. He found that in a dry atmosphere, the adiabatic lapse rate was a limit for the layer to be stable. The lapse rate inside the layer was taken by him as a constant. In the atmosphere, at high levels, often turbulence is met with over short periods. There is not much evidence of super-adiabatic lapse rates there. In addition to other factors which might have induced such turbulence, the change in the temperature height structure might also be considered as a contributory cause. One of the factors which might be changing the temperature height structure may be the radiative effects due to incursion of a different absorptive layer. (An elementary example was considered by the author in 1932).

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